Simplification d'une preuve de l'article de Muñoz

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We would like to study local existence for the Cauchy problem for

$$(PDE)\ \begin{cases} i\partial_t w + \partial_x^2 w = -[2|w|^2 + w^2 + |w|^2 w + 2Re(w)] & (t,x) \in [0,T] \times \mathbb{R},\ w(t,x) \in \mathbb{C} \\ w(0) = w_0 & x \in \mathbb{R} \end{cases}$$

This PDE is studied by Claudio Muñoz in his article "Instability in nonlinear Schrödinger breathers" [1].

We will present a proof of the following local existence theorem:

Theorem 1. *For any $T > 0$, there exists $\delta = \delta(T) > 0$ such that for any $w_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$ and $\|w_0\|_{H^s} < \delta$, there exists a solution to (PDE) $w \in C([0,T], H^s(\mathbb{R}))$.*

For any $\delta > 0$, there exists $T = T(\delta) > 0$ such that for any $w_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$ and $\|w_0\|_{H^s} < \delta$, there exists a solution to (PDE) $w \in C([0,T], H^s(\mathbb{R}))$.

Finally, we have the following alternative: if I is the maximal interval of existence of w, and $\sup I < +\infty$, then $\lim_{t\uparrow \sup I} \|w(t)\|_{H^s} = +\infty$.

Proof of the theorem

First, we will consider a linear problem:

$$(L)\ \begin{cases} i\partial_t w + \partial_x^2 w = G \\ w(0) = w_0 \end{cases}$$

where $G \in C([0,T], H^s(\mathbb{R}))$, with $s \geq 0$.

Definition 2. Let $s \geq 0$ and $T > 0$. We say that $w \in C([0,T], H^s(\mathbb{R}))$ is a solution to (PDE) if

1) $w(0) = w_0 \in H^s$

2) $w \in C([0,T], H^s(\mathbb{R}))$

3) for any $\varphi \in C_c^1(]0,T[, S(\mathbb{R}))$, we have in sense of distributions

$$< i\partial_t w + \partial_x^2 w, \varphi >=< G, \varphi >$$

By applying the Fourier transform to (L), we are able to give an explicit expression of the Fourier transform of the solution to the Cauchy problem for (L):

$$\widehat{w}(t,\xi) = e^{-i\xi^2 t}(\widehat{w_0}(\xi) - i\int_0^t e^{i\xi^2\sigma}\widehat{G}(\sigma,\xi)d\sigma)$$

Definition 3. Let $G[w] := -[2|w|^2 + w^2 + |w|^2 w + 2Re(w)]$

The following result is adapted from the corresponding result presented in [1]:

Lemma 4. *Assume $w \in H^s$, $s > \frac{1}{2}$. Then $G[w] \in H^s$, and*

$$\|G[w]\|_{H^s} \lesssim_s \|w\|_{H^s} + \|w\|_{H^s}^3$$

That allows us to have a notion of solution $w \in C([0,T], H^s)$, for $s > \frac{1}{2}$, to (PDE).

We will also need the following lemma, which is also adapted from the corresponding lemma in [1]:

Lemma 5. *Let $s > \frac{1}{2}$ and $w_1, w_2 \in H^s$. Then, we have*

$$\|G[w_1] - G[w_2]\|_{H^s} \lesssim_s (1 + \|w_1\|_{H^s}^2 + \|w_2\|_{H^s}^2)\|w_1 - w_2\|_{H^s}$$

Let us consider some parameters $\delta, T, \lambda > 0$, $C \geq 1$ and $s > \frac{1}{2}$. We assume that

$$\|w_0\|_{H^s} < \delta$$

We define the Banach space

$$\mathcal{B}(\lambda, T, C, \delta, s) := \{w \in C([0,T], H^s), \sup_{t\in[0,T]} e^{-\lambda t}\|w(t)\|_{H^s} \leq C\delta\}$$

muni de la norme $\|w\| := \sup_{t\in[0,T]} e^{-\lambda t}\|w(t)\|_{H^s}$

We define a function F on $\mathcal{B}(\lambda, T, C, \delta, s)$ such that $F(u) = w$ is the solution of (L) for $G = G[u]$.

Let us first prove the second part of the theorem. Let $\delta > 0$ and $s > \frac{1}{2}$. We shall find T, λ, C such that F is a contraction on $\mathcal{B}(\lambda, T, C, \delta, s)$. The fixed point principle allows to finish the proof.

We do the following estimation:

$$\begin{aligned}
e^{-\lambda t}\|w(t)\|_{H^s} &\lesssim_s e^{-\lambda t}\|\widehat{w}(t)\|_{L^2} + e^{-\lambda t}\||\xi|^s\widehat{w}(t)\|_{L^2} \\
&\lesssim_s e^{-\lambda t}\delta + e^{-\lambda t}\int_0^t \|G(\sigma)\|_{H^s} d\sigma \quad \text{by Minkowski inequality} \\
&\lesssim_s e^{-\lambda t}\delta + e^{-\lambda t}\int_0^t (\|u(\sigma)\|_{H^s} + \|u(\sigma)\|_{H^s}^3) d\sigma \\
&\lesssim_s e^{-\lambda t}\delta + e^{-\lambda t}\int_0^t (e^{\lambda\sigma}C\delta + e^{3\lambda\sigma}(C\delta)^3) d\sigma \\
&\lesssim_s e^{-\lambda t}\delta + tC\delta + e^{2\lambda t}t(C\delta)^3 \\
&\lesssim_s e^{-\lambda t}\delta + TC\delta + Te^{2\lambda T}(C\delta)^3
\end{aligned}$$

This proves that $e^{-\lambda t}\|w(t)\|_{H^s} \leq C\delta$ in the case we choose C big enough and then T small enough (there is no condition on λ, but we should fix it before choosing C and T).

We also do the following estimation:

$$\begin{aligned}
e^{-\lambda t}\|w_1(t)-w_2(t)\|_{H^s} &\lesssim_s e^{-\lambda t}\int_0^t \|G_1(\sigma)-G_2(\sigma)\|_{H^s} d\sigma \\
&\lesssim_s e^{-\lambda t}(1+e^{2\lambda T}(C\delta)^2)\int_0^t \|u_1(\sigma)-u_2(\sigma)\|_{H^s} d\sigma \\
&\lesssim_s e^{-\lambda t}(1+e^{2\lambda T}(C\delta)^2)T \sup_{\sigma\in[0,t]} \|u_1(\sigma)-u_2(\sigma)\|_{H^s} \\
&\lesssim_s (1+e^{2\lambda T}(C\delta)^2)T \sup_{\sigma\in[0,t]} \|u_1(\sigma)-u_2(\sigma)\|_{H^s} e^{-\lambda\sigma} \\
&\lesssim_s (1+e^{2\lambda T}(C\delta)^2)T \sup_{\sigma\in[0,T]} \|u_1(\sigma)-u_2(\sigma)\|_{H^s} e^{-\lambda\sigma}
\end{aligned}$$

Thus, it is a contraction in the case we choose T even smaller, if needed.

This proves the second part of the theorem.

Let us now prove the first part of the theorem. Let $T > 0$ and $s > \frac{1}{2}$. We shall find δ, λ, C such that F is a contraction on $\mathcal{B}(\lambda, T, C, \delta, s)$. The fixed point principle allows to finish the proof.

The estimation that we have done above for $e^{-\lambda t}\|w(t)\|_{H^s}$ is still correct in this case, but we will change a little bit the end:

$$\begin{aligned}
e^{-\lambda t}\|w(t)\|_{H^s} &\lesssim_s e^{-\lambda t}\delta + e^{-\lambda t}\int_0^t (e^{\lambda\sigma}C\delta + e^{3\lambda\sigma}(C\delta)^3)d\sigma \\
&\lesssim_s e^{-\lambda t}\delta + \frac{C\delta}{\lambda} + e^{2\lambda t}\frac{(C\delta)^3}{3\lambda} \\
&\lesssim_s \delta(1+\frac{C}{\lambda}+e^{2\lambda T}\frac{C^3\delta^2}{3\lambda})
\end{aligned}$$

Thus,

$$e^{-\lambda t}\|w(t)\|_{H^s} \leq K_s\delta(1+\frac{C}{\lambda}+e^{2\lambda T}\frac{C^3\delta^2}{3\lambda})$$

In order to have $e^{-\lambda t}\|w(t)\|_{H^s} \leq C\delta$, it suffices to choose C big enough ($C > 2K_s$) and then λ big enough (such as $\frac{C}{\lambda} < \frac{1}{4}$) and δ small enough (such as $e^{2\lambda T}\frac{C^3\delta^2}{3\lambda} < \frac{1}{4}$).

Let $A := \sup_{t\in[0,T]} e^{-\lambda t}\|u_1(t)-u_2(t)\|_{H^s}$. We would like to make the right choice of constants to have for $t \in [0,T]$, $e^{-\lambda t}\|w_1(t)-w_2(t)\|_{H^s} \leq \beta A$ with $0 < \beta < 1$. Now, we will do the following estimation:

$$
\begin{aligned}
e^{-\lambda t}\|w_1(t)-w_2(t)\|_{H^s} &\lesssim_s e^{-\lambda t}\int_0^t \|G_1(\sigma)-G_2(\sigma)\|_{H^s}d\sigma \\
&\lesssim_s e^{-\lambda t}\int_0^t (1+e^{2\lambda\sigma}(C\delta)^2)e^{\lambda\sigma}Ad\sigma \\
&\lesssim_s \frac{1-e^{-\lambda t}}{\lambda}A + e^{2\lambda T}T(C\delta)^2 A \\
&\lesssim_s (\frac{1}{\lambda}+Te^{2\lambda T}(C\delta)^2)A
\end{aligned}
$$

This is a contraction in the case we choose λ even bigger and then δ even smaller, if needed.

This proves the first part of the theorem.

To prove the last statement of the theorem, suppose that there exists $M > 0$ such as $t_n \to \sup I$ and $\forall n \in \mathbb{N}\, \|w(t_n)\|_{H^s} \leq M$. Then, by the second part of the theorem, there exists $T(M) > 0$. Take t_n such that $|t_n - \sup I| < T(M)$, then the second part of the theorem contradicts the definition of $\sup I$.