DYNAMICS OF NONLINEAR DISPERSIVE OBJECTS: STUDY OF MULTI-BREATHERS OF (mKdV)

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1. Modified Korteweg-de Vries equation

The modified Korteweg-de Vries equation is written as follows

$$\partial_t u + \partial_x(\partial_{xx} u + u^3) = 0 \qquad \text{(mKdV)}$$

where $u : (t, x) \in \mathbb{R} \times \mathbb{R} \longmapsto u(t, x) \in \mathbb{R}$.
We list some important specificities about (mKdV) equation:

- If the initial data is in H^s for $s \geq \frac{1}{4}$, then the Cauchy problem is globally well posed in $C_b(\mathbb{R}, H^s)$.
- (mKdV) is a special case of (gKdV) equations, where the cubic nonlinearity can be replaced by any degree. Another special case is (KdV) with quadratic nonlinearity.
- Like (KdV), (mKdV) is integrable. That means that we have an infinity of conservation laws. Here, we use the first three of them, that is why we reason at H^2 level.
- Unlike (KdV), and like (gKdV) equations with odd degree nonlinearity, if u is a solution of (mKdV), then $-u$ is a solution of (mKdV).
- Unlike other (gKdV) equations, (mKdV) equation has special breather solutions.

2. Solitons and antisolitons of (mKdV)

For $c > 0$, we set a regualar, positive and pair function:

$$Q_c(x) = \left(\frac{2c}{\cosh^2(\sqrt{c}x)}\right)^{\frac{1}{2}}.$$

Definition. *A* ***soliton*** *of (mKdV) is the following solution of (mKdV):* $R_c(t, x; x_0) = Q_c(x - x_0 - ct)$ *with* $x_0 \in \mathbb{R}$ *and* $c > 0$. *An* ***antisoliton*** *of (mKdV) is the following solution of (mKdV):* $R_c(t, x; x_0) = -Q_c(x - x_0 - ct)$ *with* $x_0 \in \mathbb{R}$ *and* $c > 0$.

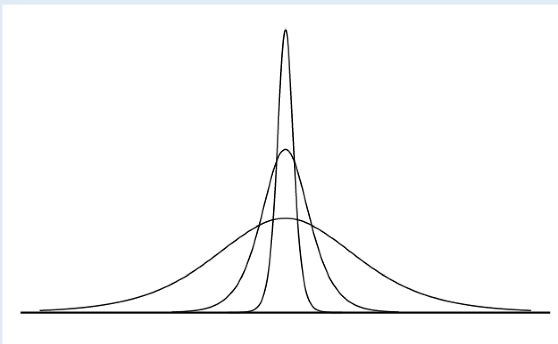

Figure 1: We see three solitons with distict velocities c, centered at the same point. The more c is large, the more the soliton is tall and narrow (and fast). A soliton moves with a constant velocity c to the right without deformation, starting from the initial position x_0.

3. Breathers of (mKdV)

Definition. *For* $\alpha, \beta \in \mathbb{R}\backslash\{0\}$ *and* $x_1, x_2 \in \mathbb{R}$, *a* ***breather*** *of (mKdV) is given by:*

$$B_{\alpha,\beta}(t, x; x_1, x_2) = 2\sqrt{2}\partial_x \left[\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\right)\right],$$

où $y_1 := x + \delta t + x_1$, $y_2 := x + \gamma t + x_2$, $\delta := \alpha^2 - 3\beta^2$ *et* $\gamma := 3\alpha^2 - \beta^2$.

We can see a breather as a function periodic in time that, in addition, propagates at a constant velocity. The velocity of propagation is $-\gamma$. Unlike a soliton, it can be positive, zero or negative.

Remarks.
We do not need to define an "antibreather", because it is enough to replace the parameter x_1 *by* $x_1 + \frac{\pi}{\alpha}$.
When $\alpha \to 0$ *with* β *fixed,* $B_{\alpha,\beta}$ *tends to another special solution of (mKdV): the* ***double-pole solution****: it is a soliton-antisoliton pair splitting up at loagarithmic rate. What we prove for breathers is not true for this limit.*

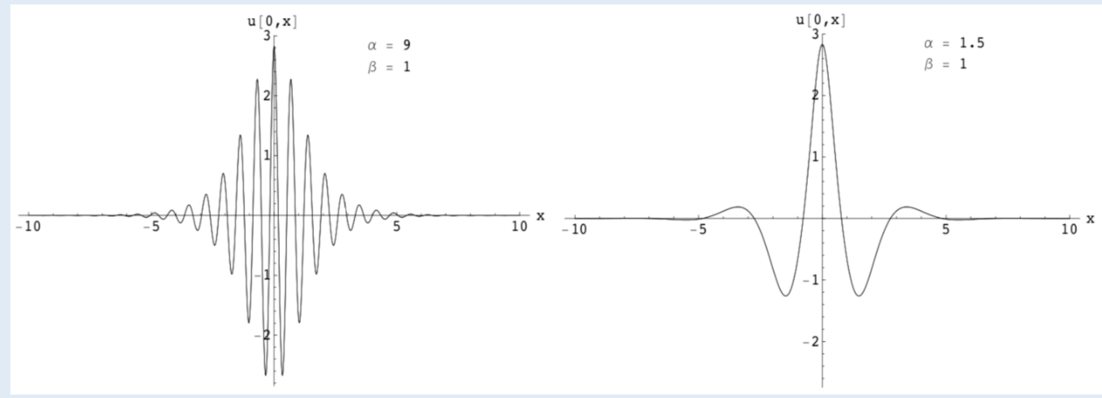

Figure 2: At the left, a breather with parameters $\alpha = 9$ and $\beta = 1$. At the right, a breather with parameters $\alpha = 1.5$ and $\beta = 1$.

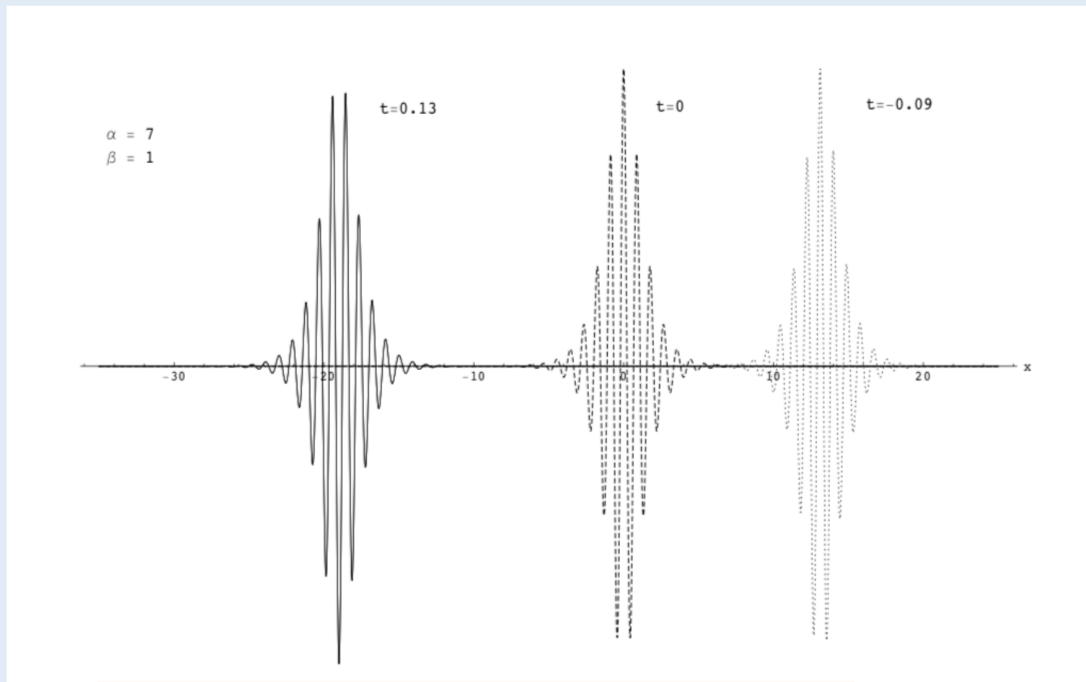

Figure 3: Evolution of a breather in time.

4. Useful conservation laws of (mKdV)

Proposition. *For* u *a solution of (mKdV), the following integrals are conserved in time:*

$$M[u] = \frac{1}{2}\int u^2, \quad E[u] = \frac{1}{2}\int u_x^2 - \frac{1}{4}\int u^4,$$

$$F[u] = \frac{1}{2}\int u_x x^2 - \frac{5}{2}\int u^2 u_x^2 + \frac{1}{4}\int u^6.$$

Proposition. *For* $c, \alpha, \beta > 0$,

$$M[R_c] = 2c^{1/2}, \quad M[B_{\alpha,\beta}] = 4\beta,$$

$$E[R_c] = -\frac{2}{3}c^{3/2}, \quad E[B_{\alpha,\beta}] = -\frac{4}{3}\beta(\beta^2 - 3\alpha^2),$$

$$F[R_c] = \frac{2}{5}c^{5/2}, \quad F[B_{\alpha,\beta}] = \frac{4}{5}\beta(\beta^4 - 10\beta^2\alpha^2 + 5\alpha^4).$$

5. Chronology of some key results about dynamics of solitons of breathers

1973 (Wadati)	Discovery of breathers by inverse scattering method [6].
1986 (Weinstein)	H^1 orbital stability of (gKdV) solitons [5].
2001 (Martel, Merle)	Asymptotic stability of (gKdV) solitons [8].
2002 (Martel, Merle, Tsai)	H^1 orbital and asymptotic stability of a sum of solitons of (gKdV) [4].
2005 (Martel)	Existence and uniqueness in H^1 of multisolitons of (gKdV) for a given set of distinct velocities. Existence in H^s for $s \geq 1$ [2].
2013 (Alejo, Muñoz)	Orbital stability in H^2 of (mKdV) breathers [1].
2019 (Chen, Liu)	Soliton-breather resolution of (mKdV) [7].

6. Presentation of the problem

We consider given $P_1, ..., P_J$ a set of solitons and breathers, and we suppose that their velocities are ordered in an increasing order $v_1 < ... < v_J$. We denote $x_1(t), ..., x_J(t)$ their positions. We are interested in their sum:

$$P(t, x) = \sum_{j=1}^{J} P_j(t, x).$$

We ask ourselves if there exists a solution p of (mKdV) such that

$$\|P(t) - p(t)\|_{H^2} \to_{t\to+\infty} 0.$$

We also ask ourselves if such a solution is unique and if such a solution satisfies a form of stability.

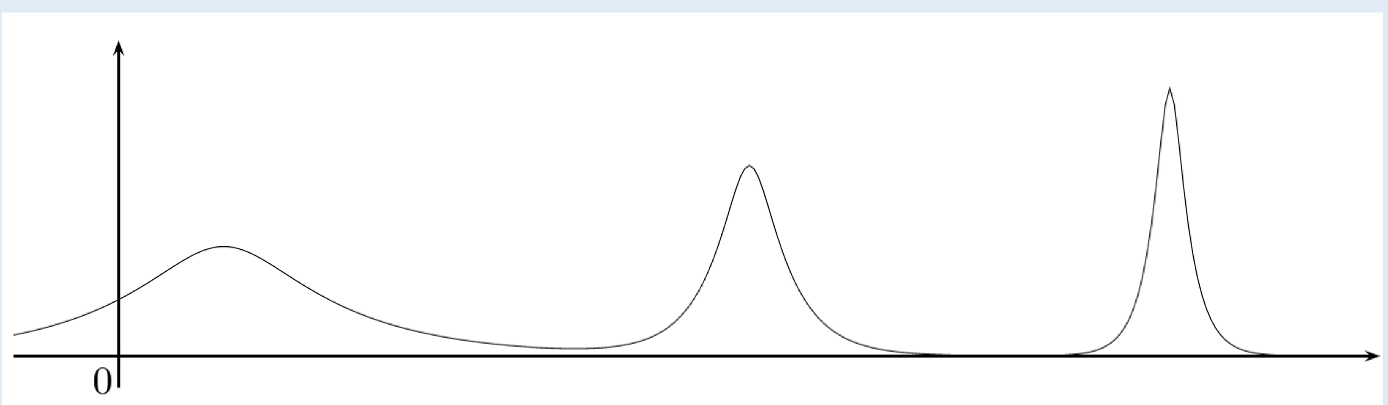

Figure 4: Example of such an object for the case of three different solitons. This case is already well studied, it is a multisoliton. We can observe that, for a time large enough, the fastest solitons are at the right and the slowest are at the left, i.e. the solitons are ordered by increasing velocity. We want to find out what happens if we add breathers to this set of objects.

In 1982, by inverse scattering method, Wadati and Okhuma have found the following explicit formula for multi-breathers:

$$p(t, x) = 2\sqrt{2}\partial_x \arctan[\Im \det(I + V)/\Re \det(I + V)],$$

where I is the identity matrix and V is a matrix defined from parameters of the considered set of solitons and breathers and can be found in [10].
This formula permits to give us a solution that is a multi-breather when $t \to +\infty$ and when $t \to -\infty$ for the same set of objects, but with different translation parameters. It also allows us to have a multi-breather for any translation parameters when $t \to +\infty$.
But, it doesn't allow us to see in which sense p tends to P when $t \to +\infty$, nor to see if the multi-breather is unique or stable. We do not use this formula in our proofs.

7. New proven results

By a suitable adaptation of ideas from [1, 2, 3, 4], we prove the following results:

Theorem 1
There $\theta > 0$, $T^* > 0$ *and* $A_s > 0$ *for any* $s \geq 2$ *such that there exists a solution* $u \in C([T^*, +\infty), H^2(\mathbb{R}))$ *of (mKdV) such that,*

$$\forall t \geq T^*, \quad \|u(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t}.$$

Moreover, if there exists $D > 0$ *such that for all* $j \geq 2$, $x_j(0) \geq x_{j-1}(0) + D$, *then* A_s, θ, T^* *do not depend on translation parameters of our objects, but only on the shape parameters and on* D.

Theorem 2
If $v_2 > 0$, *there exists* $A_0, \theta_0, D_0, a_0 > 0$ *such that the following is true. Let* $u_0 \in H^2(\mathbb{R})$, $D \geq D_0$ *and* $0 \leq a \leq a_0$, *such that*

$$\|u_0 - P(0)\|_{H^2} \leq a, \quad \text{and } x_j(0) > x_{j-1}(0) + 2D, \text{ for all } j = 2, ..., J.$$

Let $u(t)$ *be a solution of (mKdV) such that* $u(0) = u_0$. *Then, there exists translation parameters defined for any* $t \geq 0$ *such that*

$$\forall t \geq 0, \quad \|u(t) - \widetilde{P}(t)\|_{H^2} \leq A_0(a + e^{\theta_0 D}),$$

where $\widetilde{P}$ *corresponds to* P *with modified translation parameters.*

Theorem 3
If $v_2 > 0$, *there exists a unique solution* $u \in C([T^*, +\infty), H^2(\mathbb{R}))$ *of (mKdV), for some* $T^* > 0$, *such that*

$$\|u(t) - P(t)\|_{H^2} \to_{t\to+\infty} 0.$$

8. Further perspectives of research

- Asymptotic stability for breathers moving to the right and for multi-breathers constituted of solitons and breathers moving to the right, should work by adapting the arguments from [8].
- Study the double-pole solution, is it unique? It is certainly not stable, though.

9. References

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